

At most at last

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Abstract. Recent pragmatic accounts derive the ignorance inferences to which superlative modifiers give rise as quantity implicatures in a neo-Gricean framework. While these approaches successfully account for the interaction of *at least* with modals, the behavior of *at most* remains a puzzle. I propose that *at most* is composed of an antonymizing operator and *at least* and show how this accounts for the interaction of *at most* with modals and quantifiers in a neo-Gricean framework.

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1. Superlative modifiers and ignorance inferences

The superlative modifiers *at least* and *at most* have received a great deal of attention in the recent semantic and pragmatic literature (Geurts and Nouwen 2007, Büring 2008, Cummins and Katsos 2010, Nouwen 2010 and 2015, Schwarz 2011 and 2013, Cohen and Krifka 2014, Coppock and Brochhagen 2013, Kennedy 2015). What makes them a particularly interesting object of study is the fact that in most contexts, *at least* and *at most* imply speaker ignorance, i.e. they convey that the speaker isn't sure about the precise value under discussion (see Geurts and Nouwen 2007, Nouwen 2010). Sentence (1), for instance, conveys that the speaker is not sure how many beers exactly John had last night. The only thing she is sure about is that the number is not less than three. But for all she knows, John might have had four or five etc. beers.

(1) John had **at least** three beers last night.

In certain environments, however, the implication of speaker uncertainty is absent. In particular, ignorance inferences can be suppressed in certain combinations of *at least* and *at most* with modals (see Geurts and Nouwen 2007). Sentence (2), where *at least* occurs under a necessity modal, has a reading which Büring (2008) calls authoritative. Under this reading, (2) doesn't convey speaker ignorance, but rather expresses that 10 pages is the minimally required length of the paper. This reading is graphically illustrated in (2a), where '----' signifies the range of permissible paper lengths — which I will also refer to as deontic range — and 10pp is its lower bound.

- (2) The paper **has to be at least** 10 pages long. □ + *at least*
- a. '10 pages is the minimally required length of the paper'
- [-----
10pp authoritative reading
- b. 'According to what the speaker knows, the minimally required length might be 10 pages or it might be more.'
- [////////-
10pp speaker insecurity reading

Sentence (2) has another reading conveying speaker ignorance, which can be brought out by

prefixing the utterance with “I don’t know exactly. But I think ...”. Under this reading, which Buring (2008) calls speaker insecurity reading, the speaker is unsure about the minimally required length of the paper. For all she knows, the lower bound of permissible paper lengths could be 10pp or more. This reading is graphically illustrated in (2b), where ‘/////’ signifies the epistemic range, i.e. the values that for all the speaker knows might or might not be permissible. To bring out the difference between the two readings, consider a situation in which the regulations specify that only papers that are 15 or more pages long will be accepted. In this situation, the sentence is judged false under the authoritative reading (2a), whereas under the speaker insecurity reading (2b) one cannot blame the speaker for making a false statement.

Regarding the readings that are available for certain combinations of modals and superlative modifiers, I basically follow Geurts and Nouwen (2007), who argue that $\Box + \text{at least}$ and $\Diamond + \text{at most}$ have both the authoritative and the speaker insecurity reading, whereas $\Diamond + \text{at least}$ and $\Box + \text{at most}$ only allow for the speaker insecurity reading (but I will return to $\Box + \text{at most}$ in section 3.4 below). This grouping has also been confirmed in experimental studies by McNabb and Penka (2014a,b). The readings that I take to be available for the different combinations are summarized in (2) to (5).

- (3) The paper **can** be **at least** 10 pages long.
 ‘According to what the speaker knows, the maximally allowed length might be 10 pages or more.’
 -----[/////]
 10pp speaker insecurity reading only
- (4) The paper **has to** be **at most** 10 pages long.
 ‘According to what the speaker knows, the minimally required length might be 10 pages or less.’
 [/////]-----
 10pp speaker insecurity reading only
- (5) The paper **can** be **at most** 10 pages long.
 a. ‘10 pages is the maximally allowed length of the paper’
 -----]
 10pp authoritative reading
 b. ‘According to what the speaker knows, the maximally allowed length might be 10 pages or less.’
 -----[/////]
 10pp speaker insecurity reading

The ignorance implications of *at least* and *at most* and their interaction with modals are currently subject to a lot of work in semantics and pragmatics. But none of the analyses proposed so far fully accounts for the interaction of *at least* and *at most* with modals. The aim of this paper is to improve on a pragmatic account that derives ignorance inferences of superlative modifiers as quantity implicatures in a neo-Gricean framework, which is conceptually attractive and empirically well motivated. As I discuss in section 2, this approach successfully accounts for the interaction of *at least* with modals, but fails for *at most*. In section 3, I propose an analysis of *at*

most where it is decomposed into an antonymizing operator and *at least*. I show that this decomposition analysis combined with a neo-Gricean account of ignorance inferences correctly predicts that *at most* gives rise to an authoritative reading in combination with possibility modals. At the end I show that the decomposition analysis also accounts for further patterns of interaction between *at most* and quantifiers.

2. A neo-Gricean account of ignorance inferences of superlative modifiers

2.1. Ignorance inferences as quantity implicatures

Büring (2008), Cummins and Katsos (2010), Schwarz (2011, 2013) and Kennedy (2015) propose that the ignorance inferences triggered by *at least* and *at most* arise as quantity implicatures. This approach is supported by the fact that ignorance inferences of superlative modifiers show one of the hallmarks of conversational implicatures and are absent in downward entailing contexts.¹ Thus, the examples in (6), where a superlative modifier occurs in the restrictor of a universal quantifier and the antecedent of a conditional, don't convey speaker ignorance.

- (6) a. Everyone who has **at least** three children is eligible for child benefits.
b. You'll loose weight if you take in **at most** 1500 calories per day.

This approach builds on a parallel to ignorance inferences arising with disjunction, which are generally taken to be derived via Gricean reasoning. While the accounts of Büring (2008) and Cummins and Katsos (2010) take the parallel to disjunction at face value and assume that *at least* *n* is semantically equivalent to 'exactly *n* or more than *n*', Schwarz (2011, 2013) and Kennedy (2015) spell out an analysis of superlative modifiers as degree operators in a neo-Gricean framework. Here I follow Schwarz (2011), who shows that ignorance inferences of *at least* and *at most* can be derived in the same way as the ignorance implications of *or* in Sauerland's (2004) neo-Gricean system.² The essential ingredients of Schwarz' analysis are the following: In the semantics, *at least* and *at most* are analyzed as degree operators expressing non-strict comparison:

- (7) a. $\llbracket \text{at least} \rrbracket = \lambda d_d. \lambda D_{dt}. \max(D) \geq d$
b. $\llbracket \text{at most} \rrbracket = \lambda d_d. \lambda D_{dt}. \max(D) \leq d$

In the pragmatics, utterances with *at least* or *at most* trigger scalar alternatives which are the

¹ While superlative modifiers seem to be banned from the scope of negation and other negative expressions, they are more acceptable in some non-strictly negative downward entailing contexts. What causes the polarity sensitivity of superlative modifiers is an open question.

² Mayr (2013) and Schwarz (2013) note that Sauerland's algorithm needs to be revised and based on the notion of Innocent Exclusion (Fox 2007) in order to prevent the generation of unattested scalar implicatures for scalar modifiers. I neglect this issue for the purpose of this paper and circumvent the problem for Sauerland's basic algorithm by considering just those scalar alternatives that asymmetrically entail the assertion and where the number is closest to the modified numeral.

cross-product of substituting (i) the modified number by other numerals or measure phrases and (ii) *at least* and *at most* by each other and *exactly*.

$$(8) \quad \llbracket \text{exactly} \rrbracket = \lambda d_d. \lambda D_{dt}. \max(D) = d$$

With these assumptions ignorance inferences are generated for unembedded occurrences of *at least* and *at most* in Sauerland's system, where scalar implicatures and ignorance inferences are two sides of the same coin. Scalar implicatures arise if primary implicatures of the form "the speaker is not certain that ϕ ", where ϕ is a stronger scalar alternative, can be strengthened to secondary or scalar implicatures of the form "the speaker is certain that not ϕ ". Ignorance inferences arise if the stronger alternatives are symmetric, which means that they cannot simultaneously be false while the assertion is true, or putting it differently, the assertion is equivalent to the disjunction of the stronger alternatives. This is illustrated in the following for example (9), which has the LF and the truth-conditions shown in (10).

(9) The paper is **at least** 10 pages long.

- (10) a. [at least 10pp] [λd [the paper is d long]]
 b. $\max\{d: \text{long}(p,d)\} \geq 10\text{pp}$
 'The length of the paper is 10pp or more.'

The scalar alternatives of (9) correspond to the cross-product of substituting *at least* by *exactly* or *at most* and by substituting *10pp* by other paper lengths. Out of these, the alternatives that are more informative, i.e. asymmetrically entail the assertion, are the ones formed by substituting either *at least* by *exactly* or *10pp* by *11pp*.

- (11) Scalar alternatives to (9):
 The paper is *NumMod* *n* pages long. where $\text{NumMod} \in \{ \text{at least}, \text{exactly}, \text{at most} \}$
 $n \in \{ \dots, 9, 10, 11, \dots \}$

- (12) Stronger scalar alternatives:
 a. The paper is **exactly** 10 pages long. $\text{at least} \Rightarrow \text{exactly}$
 $\max\{d: \text{long}(p,d)\} = 10\text{pp}$
 b. The paper is **at least 11** pages long. $10\text{pp} \Rightarrow 11\text{pp}$
 $\max\{d: \text{long}(p,d)\} \geq 11\text{pp} \Leftrightarrow^3$
 $\max\{d: \text{long}(p,d)\} > 10\text{pp}$

Because the stronger alternatives in (12) are symmetric, i.e. one of them has to be true for the assertion to be true, none of the primary implicatures can be strengthened to secondary/scalar implicatures, because this would contradict the conjunction of the assertion and all the primary implicatures. Instead, the assertion and the primary implicatures taken together entail possibility implicatures. This is shown in (13), where using Gazdar's (1979) notation, $K\phi$ corresponds to 'the speaker knows/ believes ϕ ' and $P\phi$ to 'the speaker considers ϕ possible'.

³ The symbol \Leftrightarrow^* is used when the equivalence is based on the simplifying assumption that the relevant scale is discrete, i.e. that only full-page lengths are considered, which I make for ease of exposition.

- (13) a. Assertion: $K \max \{d: \text{long}(p,d)\} \geq 10pp$ A
 b. Primary implicatures: $\neg K \max \{d: \text{long}(p,d)\} = 10pp$ PiI1
 $\neg K \max \{d: \text{long}(p,d)\} > 10pp$ PiI2
 c. Possibility implicatures: $P \max \{d: \text{long}(p,d)\} = 10pp$ PoI1 (follows from A + PiI2)
 $P \max \{d: \text{long}(p,d)\} > 10pp$ PoI2 (follows from A + PiI1)

The primary and possibility implicatures together correspond to ignorance inferences, which are of the form $P\phi$ & $P\neg\phi$ (note that $\neg K\phi$ is equivalent to $P\neg\phi$). According to them, the speaker doesn't know whether the paper is exactly 10pp long or whether the paper is more than 10pp long. Together with the assertion, this correctly reflects the meaning of sentence (9).

- (14) Ignorance implicatures generated:
 a. $P \max \{d: \text{long}(p,d)\} = 10pp$ & $P \neg \max \{d: \text{long}(p,d)\} = 10pp$
 b. $P \max \{d: \text{long}(p,d)\} > 10pp$ & $P \neg \max \{d: \text{long}(p,d)\} > 10pp$

Ignorance inferences for unembedded occurrences of *at most* are generated in the same way, the only difference being that now the alternative with a lower numeral is symmetric to the alternative where *at most* is substituted by *exactly*.

- (15) The paper is **at most** 10 pages long.
 $\max \{d: \text{long}(p,d)\} \leq 10pp$
- (16) Stronger scalar alternatives:
 a. The paper is **exactly** 10 pages long. $\text{at most} \Rightarrow \text{exactly}$
 $\max \{d: \text{long}(p,d)\} = 10pp$
 b. The paper is at most **9** pages long. $10pp \Rightarrow 9pp$
 $\max \{d: \text{long}(p,d)\} \leq 9pp \iff^* \max \{d: \text{long}(p,d)\} < 10pp$
- (17) Ignorance implicatures generated:
 a. $P \max \{d: \text{long}(p,d)\} = 10pp$ & $P \neg \max \{d: \text{long}(p,d)\} = 10pp$
 b. $P \max \{d: \text{long}(p,d)\} < 10pp$ & $P \neg \max \{d: \text{long}(p,d)\} < 10pp$
 'The speaker doesn't know whether the paper is exactly 10pp long or whether the paper is less than 10pp long.'

The neo-Gricean analysis thus accounts for the fact that unembedded occurrences of *at least* and *at most* give rise to ignorance inferences. It also explains why numerals modified by superlative modifiers don't give rise to scalar implicatures, in contrast to bare numerals. Moreover, it makes certain predictions for the interaction of superlative modifiers with modals, which are discussed in the following sections.

2.2. Interaction with necessity modals

Since superlative modifiers are analyzed as degree operators, two different scope orders are possible when they interact with modals. If a superlative modifier is interpreted in the scope of a necessity modal, the stronger scalar alternatives in (20) are not symmetric. They can simultaneously be false while the assertion is true, namely in case of (18) if the permissible

paper length corresponds to a range including 10 and more pages. Because these alternatives are not symmetric, they give rise to the scalar implicatures in (21).

(18) The paper **has to be at least** 10 pages long.

- (19) a. has to [[at least 10pp] [λd [the paper be d long]]] $\square > at\ least$
 b. $\square \max\{d: \text{long}(p,d)\} \geq 10pp$
 ‘In all the acceptable worlds, the length of the paper is 10pp or more.’

(20) Stronger scalar alternatives:

- a. $\square \max\{d: \text{long}(p,d)\} = 10pp$ *at least \Rightarrow exactly*
 ‘In all the acceptable worlds, the length of the paper is exactly 10pp.’
 b. $\square \max\{d: \text{long}(p,d)\} \geq 11pp <==>^*$ *10pp \Rightarrow 11pp*
 $\square \max\{d: \text{long}(p,d)\} > 10pp$
 ‘In all the acceptable worlds, the paper is longer than 10pp.’

(21) Scalar implicatures generated:

- a. $K - \square \max\{d: \text{long}(p,d)\} = 10pp$
 b. $K - \square \max\{d: \text{long}(p,d)\} > 10pp$

According to these scalar implicatures, the speaker is sure that the paper doesn’t have to be exactly 10pp long and that the paper doesn’t have to be more than 10pp long. Together with the asserted content this is true iff the permissible paper lengths correspond to a range of values whose lower bound is 10pp. This corresponds to the authoritative reading in (22).

- (22) $\begin{matrix} [----- \\ 10pp \end{matrix}$ authoritative reading

If a superlative modifier takes wide scope over a necessity modal, the speaker insecurity reading results. Although the scope order *at least* $> \square$ is truth-conditionally equivalent to $\square > at\ least$ (see Heim 2000), the pragmatic reasoning is different. Because wide scope of *at least* and *exactly* in the alternatives leads to symmetric alternatives – just as in the case of unembedded occurrences, ignorance inferences rather than scalar implicatures are generated.

- (23) a. [at least 10pp] [λd [has to [the paper be d long]]] *at least $> \square$*
 b. $\max\{d: \square \text{long}(p,d)\} \geq 10pp$
 ‘The minimally required length of the paper is 10pp or more.’

(24) Stronger scalar alternatives:

- a. $\max\{d: \square \text{long}(p,d)\} = 10pp$ *at least \Rightarrow exactly*
 ‘The minimally required length of the paper is exactly 10pp.’
 b. $\max\{d: \square \text{long}(p,d)\} \geq 11pp <==>^* \max\{d: \square \text{long}(p,d)\} > 10pp$ *10pp \Rightarrow 11pp*
 ‘The minimally required length of the paper is more than 10pp.’

(25) Ignorance implicatures generated:

- a. $P \max\{d: \Box \text{long}(p,d)\} = 10pp \ \& \ P \neg \max\{d: \Box \text{long}(p,d)\} = 10pp$
 b. $P \max\{d: \Box \text{long}(p,d)\} > 10pp \ \& \ P \neg \max\{d: \Box \text{long}(p,d)\} > 10pp$

These ignorance implicatures express that the speaker is unsure about the minimally required length of the paper; she doesn't know whether it is exactly 10pp or more than 10pp. Together with the asserted content, this corresponds to the speaker insecurity reading.

- (26) $\frac{[/////////-----]}{10pp}$ speaker insecurity reading

Under the neo-Gricean account the two readings of sentences with a necessity modal and *at least* come down to a difference in scope: The authoritative reading arises if the superlative modifier is interpreted in scope of the necessity modal, and the speaker insecurity reading results from wide scope of the superlative modifier. As evidence for this scopal ambiguity, Buring (2008) observes that only the authoritative reading is available if movement of *at least* over the modal is blocked for independent reasons, in particular if *at least* is contained within a finite clause. Sentence (27) only has the authoritative reading (22), and the speaker insecurity reading (26) is absent. It thus contrasts with the minimally different (18), where *at least* occurs in an infinitival and both readings are available.

- (27) It is **required** that the paper be **at least** 10 pages long.

The derivations for *at most* are again parallel to those for *at least*, as the reader may verify for herself. If *at most* takes narrow scope under a necessity modal the stronger scalar alternatives, generated by substituting the modified number by lower values and *at most* by *exactly*, are not symmetric and the authoritative reading is derived. If *at most* takes wide scope, the stronger scalar alternatives are symmetric and the speaker insecurity reading results.

In general, ignorance inferences are obviated if a superlative modifier is interpreted in the scope of an operator that breaks symmetry. This arguably also accounts for other cases of ignorance obviation, e.g. under universal quantifiers (Schwarz, 2011) and generics (Nouwen 2010). However, we will see in the next section that a possibility modal does not break symmetry.

2.3. Interaction with possibility modals

While we just saw that a necessity modal breaks symmetry if a superlative modifier is interpreted in its scope and scalar rather than ignorance implicatures arise, a possibility modal does not break symmetry. Even if the possibility modal takes wide scope over a superlative modifier the stronger scalar alternatives cannot simultaneously be false while the assertion is true. Therefore ignorance inferences are generated. This is illustrated in the following for *at most* (the case for *at least* is again parallel).

- (28) The paper **can** be **at most** 10 pages long.
- (29) a. $\text{can} [[\text{at most } 10pp] [\lambda d [\text{the paper be } d \text{ long}]]]$ $\Diamond > \text{at most}$
 b. $\Diamond \max\{d: \text{long}(p,d)\} \leq 10pp$

‘There is an acceptable world where the length of the paper is 10pp or less.’

(30) Stronger scalar alternatives:

- a. $\Diamond \max\{d: \text{long}(p,d)\} = 10\text{pp}$ *at most* \Rightarrow *exactly*
 ‘There is an acceptable world where the length of the paper is exactly 10pp.’
 b. $\Diamond \max\{d: \text{long}(p,d)\} \leq 9\text{pp}$ $\Leftarrow \Rightarrow$ * *10pp* \Rightarrow *9pp*
 $\Diamond \max\{d: \text{long}(p,d)\} < 10\text{pp}$
 ‘There is an acceptable world where the length of the paper is less than 10pp.’

(31) Ignorance inferences generated:

- a. $P \Diamond \max\{d: \text{long}(p,d)\} = 10\text{pp} \ \& \ P \neg \Diamond \max\{d: \text{long}(p,d)\} = 10\text{pp}$
 b. $P \Diamond \max\{d: \text{long}(p,d)\} < 10\text{pp} \ \& \ P \neg \Diamond \max\{d: \text{long}(p,d)\} < 10\text{pp}$

If *at most* is interpreted in the scope of the modal as in (29) to (31) above, a reading results which is weak for several reasons. For one thing the truth conditions merely say that there is an acceptable world where the length of the paper is 10pp or less. In addition, strong ignorance inferences are generated according to which the speaker doesn’t know whether the paper can be exactly 10pp long or whether the paper can be less than 10pp long. Thus for all the speaker knows, the maximally allowed length might be 5pp or the minimally required length might be 10pp. This reading might not be detectable because there is another reading with stronger truth conditions and sensible ignorance inferences, derived from an LF where the superlative modifier takes wide scope, as shown in (32) to (35).

- (32) a. $[\text{at most } 10\text{pp}] \lambda d [\text{allowed} [\text{the paper be } d \text{ long}]]$ *at most* $>$ \Diamond
 b. $\max\{d: \Diamond \text{long}(p,d)\} \leq 10\text{pp}$
 ‘The maximally allowed length of the paper is 10pp or less.’

(33) Stronger scalar alternatives:

- a. $\max\{d: \Diamond \text{long}(p,d)\} = 10\text{pp}$ *at most* \Rightarrow *exactly*
 ‘The maximally allowed length of the paper is exactly 10pp.’
 b. $\max\{d: \Diamond \text{long}(p,d)\} \leq 9\text{pp}$ $\Leftarrow \Rightarrow$ * *10pp* \Rightarrow *9pp*
 $\max\{d: \Diamond \text{long}(p,d)\} < 10\text{pp}$
 ‘The maximally allowed length of the paper is less than 10pp.’

(34) Ignorance implicatures generated:

- a. $P \max\{d: \Diamond \text{long}(p,d)\} = 10\text{pp} \ \& \ P \neg \max\{d: \Diamond \text{long}(p,d)\} = 10\text{pp}$
 b. $P \max\{d: \Diamond \text{long}(p,d)\} < 10\text{pp} \ \& \ P \neg \max\{d: \Diamond \text{long}(p,d)\} < 10\text{pp}$

According to these ignorance inferences the speaker isn’t sure whether the maximally allowed length of the paper is exactly 10pp or whether the maximally allowed length is less than 10pp. Together with the asserted content, this corresponds to the attested speaker insecurity reading illustrated in (35).

- (35) -----/////////] speaker insecurity reading
10pp

In sum, the neo-Gricean account of ignorance inferences makes the following predictions regarding the interaction of superlative modifiers with modals: If *at least* or *at most* are

interpreted with wide scope over a necessity or possibility modal, the speaker insecurity reading is derived. The authoritative reading results from interpreting *at least* or *at most* in the scope of a necessity modal. Narrow scope under a possibility modal leads to a reading with strong ignorance inferences.

This correctly accounts for the readings observed for *at least*. As discussed in section 1, *at least* gives rise to an authoritative reading in combination with a necessity modal, but not in combination with a possibility modal. But the pattern is different for *at most*, where the authoritative reading is available in combination with a possibility modal, but doesn't seem to be available in combination with a necessity modal. Therefore the neo-Gricean analysis, while successful for *at least*, doesn't account for the interaction of *at most* and modals.

3. A compositional analysis of *at most*

3.1. Decomposing *at most*

Following the idea that negative antonyms are generally decomposed in the syntax into an antonymizing operator and the corresponding positive antonym (Heim 2006, 2008, Büring 2007, Alxatib 2013), I propose that *at most n* is decomposed into an antonymizing operator ANT and *at least n*:

$$(36) \quad \text{at most } n = [[n\text{-ANT}]_{d(dt)}\text{-at least}]_{(dt)t}$$

As meaning for *at least* I adopt the degree operator semantics proposed by Schwarz (2011) and Kennedy (2015), and repeated as (37).

$$(37) \quad [[\text{at least}]] = \lambda d_d. \lambda D_{dt}. \max(D) \geq d$$

For the semantics of the antonymizing operator ANT, it suggests itself to use Heim's (2006) operator *little*, defined in (38) and expressing degree negation (see also Alxatib 2013).

$$(38) \quad [[\text{little}]] = \lambda d_d. \lambda D_{dt}. \neg D(d)$$

With this meaning of *little*, however, *at most n* cannot be decomposed into *little* plus *at least n*, but would rather correspond to *little* plus *more than n*. But there is in fact independent evidence that a revision of Heim's definition of *little* is needed (see also Beck 2012). This comes from sentences like (39), where *that* serves as direct degree argument of *weigh* and anaphorically picks up the measure phrase 40kg. With the definition of *little* in (38) the truth conditions in (40b) are derived, according to which Sue weighs less than 40kg. Since (39) is intuitively perfectly compatible with Sue weighing exactly 40kg, this is not what the sentence means.

$$(39) \quad [\text{Mary only weighs 40kg.}] \text{ Sue weighs that little too.}$$

$$(40) \quad \text{a. } [\text{that little}] [\lambda d [\text{Sue weighs } d\text{-much}]]$$

$$\text{b. } \neg[\text{WEIGHT}(s) \geq 40\text{kg}] = \text{WEIGHT}(s) < 40\text{kg}$$

In order to derive the correct meaning for sentence (39), we need a definition of *little* in which only higher degrees are negated, but not the degree contributed by the first argument:

$$(41) \quad \llbracket \text{little}_2 \rrbracket = \lambda d_d. \lambda D_{dt}. \forall d' > d: \neg D(d')$$

With this we derive the meaning (42b), according to which Sue doesn't weigh more than 40kg. This can then be strengthened by scalar implicature to mean that Sue's weight is exactly 40kg, just as for the sentence *Sue weighs 40kg*. This correctly captures the meaning of (39).

$$(42) \quad \begin{array}{l} \text{a. } [\text{that little}_2] [\lambda d [\text{Sue weighs } d\text{-much}]] \\ \text{b. } \forall d' > 40\text{kg}: \neg[\text{WEIGHT}(s) \geq d'] = \\ \quad \neg[\text{WEIGHT}(s) > 40\text{kg}] \end{array}$$

I thus use this revised definition of *little* as the meaning of the antonymizing operator ANT. As noted by Beck (2012), this renders the antonymizing operator ANT equivalent to the straightforward definition of *at most* repeated from (7b) above.

$$(43) \quad \llbracket \text{ANT} \rrbracket = \lambda d_d. \lambda D_{dt}. \forall d' > d: \neg D(d')$$

$$(7b) \quad \llbracket \text{at most} \rrbracket = \lambda d_d. \lambda D_{dt}. \max(D) \leq d$$

3.2. Alternatives and ignorance inferences of *at most*

With these assumptions about the syntax-semantics interface, let us now turn to the question what consequences the compositional analysis of *at most* has for the pragmatics. In particular the question needs to be addressed what the scalar alternatives are for an utterance with *at most*. In this respect I follow Katzir (2007) and Fox & Katzir (2011), who argue that alternatives are structurally defined and generated by substitution of lexical categories and deletion. In particular, I assume that the scalar alternatives for an utterance with *at most* are generated by (i) substituting numerals or measure phrases by each other; (ii) substituting *at least* by *exactly* (see Schwarz 2011) and (iii) deleting ANT (see Alxatib 2013). In addition, I adopt the common assumption that (iv) modals are substituted in the alternatives.

With these assumptions about the meaning of *at most* and scalar alternatives, ignorance inferences for unembedded occurrences of *at most* are generated in the same way as in the neo-Gricean account: the stronger scalar alternatives (47) are the same as under Schwarz' (2011) analysis. Since these are symmetric, ignorance inferences are generated.

$$(44) \quad \text{The paper is } \mathbf{at\ most} \text{ } 10 \text{ pages long.}$$

$$(45) \quad \begin{array}{l} \text{a. } \text{ANT-10pp} [\lambda d_2 [\text{at least-}d_2 [\lambda d_1 [\text{the paper is } d_1\text{-long}]]]] \\ \text{b. } \forall d' > 10\text{pp}: \neg [\max\{d: \text{long}(p,d)\} \geq d] \iff \\ \quad \neg \max\{d: \text{long}(p,d)\} > 10\text{pp} \end{array}$$

(52) Scalar alternatives:

The paper is *Pol Mod NumMod* *n* pages long.

where $Pol \in \{\text{ANT}, \emptyset\}$

$Mod \in \{\text{allowed}, \text{required}\}$

$NumMod \in \{\text{at least}, \text{exactly}\}$

$n \in \{\dots, 9, 10, 11, \dots\}$

We now have to consider eight scalar alternatives. It turns out that out of these, only the two shown in (53) asymmetrically entail the assertion.⁴ Crucially, the alternative (53a), generated by substituting the measure phrase with a lower value, doesn't have a symmetric counterpart. This is due to the fact that ANT, which has the semantics attributed by Schwarz (2011) to *at most*, can be deleted in the alternatives but not substituted by *exactly*. The alternative (53a) thus leads to a scalar implicature (54a), according to which the speaker is sure that the paper can be more than 9pp long. In addition, the alternative (53b) also leads to a scalar implicature (54b), according to which the speaker is sure that the paper doesn't have to be exactly 10pp long.

(53) Stronger scalar alternatives:

a. $\neg \Diamond \max\{d: \text{long}(p,d)\} > 9pp$

$10pp \Rightarrow 9pp$

b. $\Box \max\{d: \text{long}(p,d)\} = 10pp$

$\text{ANT}, \Diamond \Rightarrow \Box, \text{at least} \Rightarrow \text{exactly}$

(54) Scalar implicatures generated:

a. $K \Diamond \max\{d: \text{long}(p,d)\} > 9pp$

b. $K \neg \Box \max\{d: \text{long}(p,d)\} = 10pp$

Taken together, the assertion and the scalar implicatures express that the permissible paper lengths correspond to a range of values whose upper bound is 10pp. This corresponds to the authoritative reading (55), which is in fact the most salient interpretation for sentence (49).

(55)] authoritative reading
10pp

The decompositional analysis can thus derive the authoritative reading for the combination of *at most* with a possibility modal, which other analyses fail to account for. In addition to the authoritative reading, the other two readings that the basic neo-Gricean approach discussed in section 2 derives are also generated from the other two available LFs (50a) and (50b). In general, if ANT and *at least* take adjacent scope, the same truth conditions and ignorance inferences are derived as under the non-decompositional analysis of Schwarz (2011). If both ANT and *at least* are interpreted in the scope of the possibility modal as in (50a), strong ignorance inferences are derived. The speaker insecurity reading is derived from the LF (50b), where both ANT and *at least* take scope under the possibility modal.

3.4. Interaction of *at most* with necessity modals

To complete the discussion of the predictions of the decompositional analysis, we also need to reconsider the interaction of *at most* with a necessity modal, as in (56). Again, the three scope

⁴ In fact, the alternative derived by substituting *at least* by *exactly* and *10pp* by *9pp* is equivalent to (53a). In case of equivalent alternatives, I only consider the one requiring the fewest changes from the utterance.

orders in (57) have to be considered.

(56) The paper **has to** be **at most** 10 pages long.

- (57) a. has to [ANT-10 [λd_2 [at least- d_2 [λd_1 [the paper be d_1 -long]]]]] $\Box > \text{ANT} > \text{at least}$
 b. ANT-10 [λd_2 [at least- d_2 [λd_1 [has to [the paper be d_1 -long]]]]] $\text{ANT} > \text{at least} > \Box$
 c. ANT-10 [λd_2 [has to [at least- d_2 [λd_1 [the paper be d_1 -long]]]]] $\text{ANT} > \Box > \text{at least}$

In the LF (57c), which is available in the decompositional, but not the basic neo-Gricean analysis, the necessity modal takes scope in between ANT and *at least*. This scope order leads to the symmetric alternatives (59a) and (59b), and thus to strong ignorance inferences. As before, I assume that the strong ignorance reading is masked by the existence of the speaker insecurity reading with sensible ignorance inferences.

- (58) a. ANT-10pp [λd_2 [has to [at least- d_2 [λd_1 [the paper be d_1 -long]]]]] $\text{ANT} > \Box > \text{at least}$
 b. $\forall d' > 10\text{pp}$: $\neg \Box \max\{d: \text{long}(p,d)\} \geq d \iff \neg \Box \max\{d: \text{long}(p,d)\} > 10\text{pp}$
 'The paper doesn't have to be more than 10pp long.'

(59) Stronger scalar alternatives:

- a. $\neg \Box \max\{d: \text{long}(p,d)\} > 9\text{pp}$ $10\text{pp} \Rightarrow 9\text{pp}$
 b. $\Diamond \max\{d: \text{long}(p,d)\} = 10\text{pp}$ $\text{ANT}, \Box \Rightarrow \Diamond, \text{at least} \Rightarrow \text{exactly}$
 c. $\neg \Diamond \max\{d: \text{long}(p,d)\} > 10\text{pp}$ $\Box \Rightarrow \Diamond$

(60) Ignorance inferences generated:

- a. $P \neg \Box \max\{d: \text{long}(p,d)\} > 9\text{pp} \ \& \ P \Box \max\{d: \text{long}(p,d)\} > 9\text{pp}$
 b. $P \Diamond \max\{d: \text{long}(p,d)\} = 10\text{pp} \ \& \ P \neg \Diamond \max\{d: \text{long}(p,d)\} = 10\text{pp}$
 'The speaker isn't sure whether the paper is required to be longer than 9pp and she isn't sure whether the paper is allowed to be exactly 10pp long.'

(61) Scalar implicature generated:

- $K \Diamond \max\{d: \text{long}(p,d)\} > 10\text{pp}$
 'The speaker is sure that the paper is allowed to be longer than 10pp.'

The readings derived for the two LFs (57a) and (57b) where ANT and *at least* take adjacent scope over or under the modal are again equivalent to the ones for non-decomposed *at most*. Wide scope of ANT and *at least* results in the speaker insecurity reading, and the authoritative reading is derived if both ANT and *at least* are interpreted in the scope of the necessity modal.

The decompositional analysis inherits from the neo-Gricean account the prediction that for the combination of *at most* with a necessity modal both the speaker insecurity and the authoritative reading is available. As discussed in section 1, however, only the speaker insecurity reading seems to be possible for sentence (56). Note however, that the authoritative reading is readily available if *at most* is embedded in a finite clause under a necessity modal. The naturally occurring sentences in (62) don't express speaker ignorance, but rather report or set the upper

bound of the range of permissible values:

- (62) a. [I am looking for suggestions for a dorm room microwave for my son.]
The college **requires** that it be **at most** 1 cu feet in volume and **at most** 800 Watts.⁵
'1 cu feet is the maximally allowed volume and 800 Watts is the maximally allowed power.'
b. This algorithm **requires** that variables be used **at most** once.⁶
'The maximally allowed number of variable uses is one.'

Data like (62) suggest that the authoritative reading is in fact available (and the only reading possible) if there is no choice but to interpret the modal with widest scope. Thus, I take it to be a welcome prediction that wide scope of a necessity modal results in the authoritative reading. The question remains, however, why the authoritative reading doesn't seem to be available when *at most* is contained within an infinitival complement of a necessity modal as in (56). A way to explain this could be to relate the scopal interaction of modals vis-à-vis the antonymizing operator to the one they show vis-à-vis negation. The modal verb *have to* in (56) is known to take narrow scope with respect to negation (see Iatridou and Zeijlstra 2013). This would also predict that it takes narrow scope with respect to ANT, making the authoritative reading unavailable. In order to see whether this explanation is on the right track, further types of modals and other constructions arguably involving ANT or *little* would have to be considered.⁷ I leave this issue for future research and discuss in the remainder of the paper further aspects of the interaction of *at most* with possibility modals and quantifiers.

4. Further predictions of the analysis

4.1. Scope trapping

In the analysis I propose, the authoritative reading of *at most* plus possibility modal is derived from an LF where ANT takes wide scope. This leads to the prediction that the authoritative reading shouldn't be available if movement out of the scope of the modal is blocked for independent reasons. Evidence that this prediction is borne out comes from sentences like (63) where *at most* is embedded in a finite clause. While the sentence is less than perfect and hard to interpret, it seems clear that it doesn't have the authoritative reading, according to which 10pp is the maximally allowed length of the paper.

- (63) It is **permitted** that the paper is **at most** 10 pages long.

⁵ <http://www.bogleheads.org/forum/viewtopic.php?f=11&t=100975> (accessed 30 June 2014).

⁶ <http://homepages.inf.ed.ac.uk/wadler/papers/oncetech/oncetech.ps> (accessed 30 June 2014)

⁷ One prediction would be that *at most* should readily combine with *must*, which takes wide scope with respect to negation, to yield the authoritative reading. But while the authoritative reading seems to be more easily available when *at most* is combined with *must* than when it is combined with *have to*, the combination doesn't seem to be particularly natural. This might have to do with the fact that the authoritative reading is the same as expressed by the combination of *at most* with a possibility modal. Since both possibility modals and *at most* are naturally used to express an upper bound, this combination might seem more natural than combining *at most* with necessity modals, which naturally express lower bounds (see Breheny 2008).

The absence of the authoritative reading if wide scope of the modal is enforced also provides an argument against attributing obviation of ignorance inferences under possibility modals to a Free Choice effect (see Coppock & Brochhagen (2013) and Nouwen (2015) for proposals in this direction). Current analyses of Free Choice effects arising with disjunction and indefinites (Fox 2007 among others) derive Free Choice permission readings from an LF where the possibility modal takes wide scope. If the authoritative reading of *at most* was due to a Free Choice effect, then the fact that (63) doesn't have the authoritative reading is unexpected. In the next subsection, I present further evidence for the assumption that the authoritative reading of *at most* with an existential operator is due to some meaning component of *at most* taking wide scope.

4.2. Interaction with quantifiers

Interestingly, the difference in readings between *at least* and *at most* observed in section 1 is restricted to the interaction with modals. In their interaction with non-modal existential and universal operators, *at least* and *at most* behave parallel. Both *at least* and *at most* obviate ignorance inferences under universal quantifiers. The sentences in (64) and (65) don't convey that the speaker is uncertain about the length of the papers on the reading list, but rather express that the papers are of different length with the shortest (or longest in case of *at most*) paper being exactly 10pp long. This reading is parallel to the authoritative reading arising for the combination of a superlative modifier and a universal modal, the difference being that now the distribution of values is across individual papers rather than worlds.

- (64) **Every** paper on the reading list is **at least** 10 pages long.
 ~> Some paper(s) on the reading list are exactly 10 pages long.
 ~> Some paper(s) on the reading list are more than 10 pages long.
- (65) **Every** paper on the reading list is **at most** 10 pages long.
 ~> Some paper(s) on the reading list are exactly 10 pages long.
 ~> Some paper(s) on the reading list are less than 10 pages long.

Under (singular) existential quantifiers, neither *at least* nor *at most* can be used to specify a range of values. Instead, both superlative modifiers lead to a reading with strong ignorance inferences as illustrated in (66) and (67).

- (66) **Some** paper on the reading list is **at least** 10 pages long.
 ~> The speaker doesn't know whether some paper is exactly 10 pages long.
 ~> The speaker doesn't know whether some paper is more than 10 pages long.
- (67) **Some** paper on the reading list is **at most** 10 pages long.
 ~> The speaker doesn't know whether some paper is exactly 10 pages long.
 ~> The speaker doesn't know whether some paper is less than 10 pages long.

Under the compositional analysis of *at most* this difference in the interaction with modals and quantifiers is in fact predicted. It follows from the Heim-Kennedy-generalization (Heim 2000), according to which degree operators can QR over modals, but not quantifiers. Thus *at*

least as well as ANT obligatorily take narrow scope in (64) to (67). As predicted by the neo-Gricean account, narrow scope under a universal quantifier (or modal) results in scalar implicatures expressing a range of values, while narrow scope under an existential quantifier (or modal) leads to strong ignorance inferences. (68) summarizes the pattern of readings predicted by the neo-Gricean account augmented with the decomposition analysis of *at most*.

- (68) a. $\Box/\forall > (\text{ANT} >) \text{ at least}$: authoritative/range reading
 b. $\Diamond/\exists > (\text{ANT} >) \text{ at least}$: strong ignorance reading
 c. $(\text{ANT} >) \text{ at least} > \Box/\Diamond$: speaker insecurity reading
 d. $(\text{ANT} >) \text{ at least} > \forall/\exists$: —
 e. $\text{ANT} > \forall/\exists > \text{ at least}$: —
 f. $\text{ANT} > \Diamond > \text{ at least}$: authoritative/range reading
 g. $\text{ANT} > \Box > \text{ at least}$: strong ignorance reading

The fact that *at most* gives rise to an authoritative or range reading in combination with a possibility modal but not an existential quantifier provides further support for the assumption that this reading is due to some meaning component of *at most* taking wide scope. Moreover, the fact that *at most* readily gives rise to a range reading in combination with a universal quantifier further supports the assumption that wide scope of a universal quantifier or modal obviates ignorance inferences, but that this reading may only surface if wide scope is enforced.

5. Concluding remarks

This paper started from the neo-Gricean approach to ignorance inferences of superlative modifiers, which successfully accounts for the interaction of *at least* with modals, but fails for *at most*. Crucially, *at most* gives rise to an authoritative reading in combination with a possibility modal, which the analyses of Schwarz (2011, 2013) and Kennedy (2015) cannot account for. I argued that the authoritative reading can be derived if the neo-Gricean approach is supplemented with the assumption that *at most* is decomposed into an antonymizing operator and *at least*. I also argued that this analysis explains the otherwise puzzling fact that *at least* and *at most* behave parallel in their interaction with quantifiers, but not modals.

The fact that decomposing *at most* successfully accounts for the interaction of *at most* with possibility modals and quantifiers can be taken as further support for the idea that negative antonyms are generally decomposed in the syntax (Büring 2007). But the analysis also raises the question what triggers this decomposition, or putting it differently, which element of a pair of antonyms should be analysed as the negative one. For the antonym pair *at least* — *at most* it seems that semantic rather than morphological properties are decisive. The evidence from the interaction with modals suggests that it is *at most* which is more complex and composed of the antonymizing operator, rather than *at least*, although the former is morphologically based on the positive form *much* and the latter on the negative form *little*. It might seem that *at least* is the more likely candidate for decomposition, since it already contains *little*, which is semantically equivalent to the antonymizing operator. Instead, in the analysis I propose to account for the readings of superlative modifiers, it is the downward monotonic modifier *at most* that involves the antonymizing operator.

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